Inhomogeneous "longitudinal" circularly-polarized plane waves in anisotropic elastic crystals

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Abstract

Conditions on the elastic stiffnesses of anisotropic crystals are derived such that circularly polarized longitudinal inhomogeneous plane waves with an isotropic slowness bivector may propagate for any given direction of the normal to the sagittal plane. Once this direction is chosen, then the wave speed, the direction of propagation, and the direction of attenuation are expressed in terms of the mass density, the elastic stiffnesses, and the angle between the normal to the sagittal plane and the normals (also called "optic axes") to the planes of central circular section of a certain ellipsoid. In the special case where this angle is zero, and in this special case only, such waves cannot propagate.

1 Introduction

In classical linearized elasticity theory, a special role is played by infinitesimal longitudinal homogeneous plane waves. For such waves the amplitude vector is parallel to the propagation direction $\mathbf{n}$ (say) so that all the particles oscillate along that direction $\mathbf{n}$ and the motion is one-dimensional. Such waves may propagate in every direction in an isotropic compressible elastic material, but this is no longer the case for elastic anisotropic materials such as crystals. Possible directions of propagation of longitudinal homogeneous plane waves, called "specific directions" by Borgnis [1] may be as few as three in an elastic anisotropic crystal. By assuming certain restrictions on the elastic constants, Hadamard [2] created a special model anisotropic

elastic material in which longitudinal plane waves may propagate in every direction.

Here, we consider *inhomogeneous* plane waves. These waves are attenuated in a direction different from the propagation direction. They may be described in terms of bivectors [3] – complex vectors – the amplitude bivector, $\mathbf{A}$ (say), and the slowness bivector, $\mathbf{S}$ (say), which may be written [4] $\mathbf{S} = N\mathbf{C}$, where the "directional bivector" $\mathbf{C}$ is written $\mathbf{C} = m\hat{\mathbf{m}} + i\hat{\mathbf{n}}$ ($\hat{\mathbf{m}} \cdot \hat{\mathbf{n}} = 0$, $m \geq 1$, $|\hat{\mathbf{m}}| = |\hat{\mathbf{n}}| = 1$) and N is called the "complex scalar slowness". Once the directional bivector $\mathbf{C}$ is prescribed, the slowness $\mathbf{S}$ and amplitude $\mathbf{A}$ are determined from the equations of motion. To prescribe $\mathbf{C}$ is equivalent to prescribing an ellipse with major semi-axis $m\hat{\mathbf{m}}$ and minor semi-axis $\hat{\mathbf{n}}$; this so-called "directional ellipse" for inhomogeneous plane waves is the analogue to the direction of propagation $\mathbf{n}$ for homogeneous plane waves [4]. The inhomogeneous plane wave is said to be "longitudinal" if $\mathbf{A}$ and $\mathbf{S}$ (and therefore also $\mathbf{C}$) are "parallel": $\mathbf{A} \wedge \mathbf{S} = \mathbf{0}$. What this means is that [3] the ellipses of $\mathbf{A}$ and of $\mathbf{S}$ (and $\mathbf{C}$) are all "parallel", being similar – same aspect ratio – and being similarly situated – parallel major axes and parallel minor axes. In particular we consider "circularly polarized longitudinal inhomogeneous plane waves" (CPLIPW). For such waves both $\mathbf{C}$ and $\mathbf{A}$ are isotropic, that is $\mathbf{C} \cdot \mathbf{C} = 0$, $\mathbf{A} \cdot \mathbf{A} = 0$; the ellipses corresponding to $\mathbf{C}$ and $\mathbf{A}$ are coplanar circles, the normal to the plane being $\mathbf{a} = \hat{\mathbf{m}} \wedge \hat{\mathbf{n}}$.

Here we seek to determine restrictions on the elastic constants such that for any choice of $\mathbf{a}$, the normal to the plane of $\mathbf{C}$ and $\mathbf{A}$ (which is the plane of the motion), a CPLIPW may propagate. We call the corresponding materials "special".

Starting with the constitutive equation for a general anisotropic elastic crystal (which involves twenty-one independent elastic constants), we obtain necessary and sufficient conditions on the constants in order that CPLIPWs may propagate in every plane. It turns out that nine linear relations among the elastic constants must be satisfied so that the special model material has at most twelve independent elastic constants. For such materials we determine the general structure of the (symmetric) acoustical tensor. The complex slowness N of the CPLIPW is determined for all choices of isotropic $\mathbf{C}$. Because the wave is circularly polarized, one eigenvalue of the acoustical tensor is double [4]. The remaining simple eigenvalue corresponds to a "transverse" inhomogeneous plane wave that is transverse in the sense that its amplitude bivector $\mathbf{B}$ (say) is "orthogonal" to $\mathbf{C}$: $\mathbf{B} \cdot \mathbf{C} = 0$, which means that the orthogonal projection of the ellipse of $\mathbf{B}$ upon the plane of $\mathbf{C}$ is a

circle.

The equation giving the complex slowness N for the CPLIPW is of precisely the same structure as the equation giving the complex slowness of the transverse wave. Both are of the form $\mathbf{C} \cdot \mathbf{\Theta}\mathbf{C} = N^{-2}$, $\mathbf{C} \cdot \mathbf{C} = 0$, where $\mathbf{\Theta}$ is a real symmetric tensor. An ellipsoid E (say) may be associated with $\mathbf{\Theta}$ [3]. It is seen that the slowness bivectors are obtained by first determining the central ellipsoidal section $\mathcal{E}$ (say) of the ellipsoid E by the central plane with normal $\mathbf{a}$. Then $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ are chosen to lie along the principal axes of $\mathcal{E}$. This fixes $\mathbf{C}$, and then N, and therefore $\mathbf{S}$ $(= N\mathbf{C})$ is determined. For choices of $\mathbf{a}$ along the normals to the planes of central circular sections of the ellipsoid E, there is no propagating wave.

Finally, we briefly consider the possibility of having CPLIPWs propagating in crystals of various classes: triclinic, monoclinic, etc. It is seen that CPLIPWs may not propagate in trigonal, tetragonal, and cubic crystals, nor in isotropic materials. It is seen that they may propagate in triclinic, monoclinic, orthorhombic, and hexagonal crystals, provided the linear relations among the elastic constants evoked above (or their specialization to those classes of symmetry) are satisfied. As kindly pointed out by a referee, we must ensure that the crystals are purely elastic so that mechanical fields are not coupled to electrical fields. Hence in what follows, we restrict our attention to the following crystal classes: $\bar{1}$, 2/m, mmm, $\bar{3}$m, 4/m, 4/mmm, 6/m, 6/mmm, 432, m3, m3m.

2 Basic equations

The constitutive equations relating stress components σ_{ij} with strain components e_{ij} for a homogeneous anisotropic elastic crystal are given by Hooke's law:

$$\sigma_{ij} = d_{ijkl}e_{kl}. \tag{2.1}$$

Here d_{ijkl} the elastic constants, or stiffnesses, have the symmetries,

$$d_{ijkl} = d_{jikl} = d_{klij}, \tag{2.2}$$

so that there are at most 21 independent stiffnesses. Also,

$$2e_{ij} := \partial u_i/\partial x_j + \partial u_j/\partial x_i, \tag{2.3}$$

where u_i are the displacement components: $u_i := x_i - X_i$. Here $\mathbf{x}$ is the current position of a particle initially at $\mathbf{X}$.

The equations of motion, in the absence of body forces, are given by

$$d_{ijkl}\partial^2 u_k/\partial x_l \partial x_j = \rho \partial^2 u_i/\partial t^2, \tag{2.4}$$

where ρ is the mass density of the crystal.

We consider displacements of the form

$$\mathbf{u} = \{\mathbf{A}\mathrm{e}^{i\omega(\mathbf{S}\cdot\mathbf{x}-t)}\}^+ = \mathrm{e}^{-\omega\mathbf{S}^-\cdot\mathbf{x}}\{\mathbf{A}^+\cos\omega(\mathbf{S}^+\cdot\mathbf{x}-t) - \mathbf{A}^-\sin\omega(\mathbf{S}^+\cdot\mathbf{x}-t)\}, \tag{2.5}$$

where $\mathbf{S} = \mathbf{S}^+ + i\mathbf{S}^-$ is the slowness bivector (complex vector) [4, 3] and $\mathbf{A} = \mathbf{A}^+ + i\mathbf{A}^-$ is the amplitude bivector (complex vector). When $\mathbf{A}$ is parallel to $\mathbf{S}$, that is when $\mathbf{A} = \alpha\mathbf{S}$, where α is some complex number, the inhomogeneous wave is said to be "longitudinal" [5, 6]. When $\mathbf{A}$ is isotropic, that is when $\mathbf{A}\cdot\mathbf{A} = 0$, the wave is circularly polarized, the ellipse associated with the bivector $\mathbf{A}$ being a circle [3].

Circularly polarized longitudinal inhomogeneous waves are those waves for which $\mathbf{A}$ and $\mathbf{S}$ are both isotropic and parallel. For such waves,

$$\mathbf{S}\cdot\mathbf{S} = 0, \tag{2.6}$$

or equivalently,

$$\mathbf{S}^+\cdot\mathbf{S}^+ = \mathbf{S}^-\cdot\mathbf{S}^-, \quad \mathbf{S}^+\cdot\mathbf{S}^- = 0. \tag{2.7}$$

Thus, the planes of constant phase are orthogonal to the planes of constant amplitude; the waves propagate in the direction of $\mathbf{S}^+$ whilst the amplitude decays in the direction of $\mathbf{S}^-$. The particle paths are circles in the plane of $\mathbf{A}^+$ and $\mathbf{A}^-$, or equivalently, because $\mathbf{A} = \alpha\mathbf{S}$, in the plane of $\mathbf{S}^+$ and $\mathbf{S}^-$. The sense of description of the circle is from $\mathbf{S}^+$ towards $\mathbf{S}^-$, retrograde, similar to the sense of Rayleigh waves propagating close to the free surface of a semi-infinite isotropic elastic material.

Inserting (2.5) into (2.4) gives the *propagation condition*,

$$Q_{ik}(\mathbf{S})A_k = \rho A_i, \quad Q_{ik}(\mathbf{S}) = d_{ijkl}S_jS_l, \tag{2.8}$$

where $Q_{ik}(\mathbf{S})$ is the *acoustical tensor* corresponding to the slowness bivector $\mathbf{S}$. A systematic procedure for obtaining all solutions $\mathbf{A}$, $\mathbf{S}$ of (2.8) has been introduced by Hayes [4] and is called the "directional ellipse method" or "DE-method". It consists in writing $\mathbf{S}$ as

$$\mathbf{S} = N\mathbf{C}, \tag{2.9}$$

where N is a complex number and $\mathbf{C}$ is any bivector of the form $\mathbf{C} = m\hat{\mathbf{m}} + i\hat{\mathbf{n}}$, with $\hat{\mathbf{m}}$, $\hat{\mathbf{n}}$, two unit orthogonal vectors and $m \geq 1$. We call N the *complex scalar slowness* and $\mathbf{C}$ the *directional bivector*. Inserting (2.9) into (2.8) yields the eigenvalue problem,

$$Q_{ik}(\mathbf{C})A_k = \rho N^{-2} A_i, \tag{2.10}$$

for the complex symmetric tensor $Q_{ik}(\mathbf{C})$. All solutions of the propagation condition may then be obtained by prescribing $\mathbf{C}$ arbitrarily and solving (2.10) for N^{-2} and $\mathbf{A}$, which gives the slowness bivector through (2.9), and the amplitude bivector $\mathbf{A}$ (up to a complex scalar factor). When, for some eigenvalue N^{-2}, the corresponding eigenbivector $\mathbf{A}$ is parallel to the directional bivector $\mathbf{C}$, the corresponding inhomogeneous plane wave is said to be longitudinal. Thus, for longitudinal inhomogeneous plane waves the propagation condition becomes

$$Q_{ik}(\mathbf{C})C_k = \rho N^{-2} C_i, \tag{2.11}$$

for some N^{-2}. Hence, these waves are only possible for directional bivectors $\mathbf{C}$ such that $\mathbf{Q}(\mathbf{C})\mathbf{C}$ is parallel to $\mathbf{C}$, or equivalently,

$$\mathbf{C} \times \mathbf{Q}(\mathbf{C})\mathbf{C} = \mathbf{0}, \tag{2.12}$$

or

$$\frac{Q_{1k}(\mathbf{C})C_k}{C_1} = \frac{Q_{2k}(\mathbf{C})C_k}{C_2} = \frac{Q_{3k}(\mathbf{C})C_k}{C_3}. \tag{2.13}$$

Here we seek particular classes of materials which are such that longitudinal inhomogeneous plane waves are possible for any choice of $\mathbf{C}$ satisfying $\mathbf{C} \cdot \mathbf{C} = 0$ (so that $\mathbf{C}$ is of the form $\mathbf{C} = \hat{\mathbf{m}} + i\hat{\mathbf{n}}$). That is, we wish to determine under which conditions on the stiffnesses d_{ijkl} are CPLIPWs possible for all choices of the plane of $\mathbf{C}$, or equivalently of the normal $\mathbf{a} = \hat{\mathbf{m}} \times \hat{\mathbf{n}}$.

3 Crystals for which longitudinal circularly polarized waves are possible in all planes

3.1 Necessary and sufficient conditions

Here we seek under which conditions on the stiffnesses d_{ijkl}, we have: $\mathbf{C} \times \mathbf{Q}(\mathbf{C})\mathbf{C} = \mathbf{0}$ for all $\mathbf{C} = \hat{\mathbf{m}} + i\hat{\mathbf{n}}$, or equivalently, for all $\mathbf{C}$ satisfying $\mathbf{C} \cdot \mathbf{C} = 0$.

For convenience we adopt the Voigt [7] contracted notation for the elastic stiffnesses,

$$d_{12} = d_{1122}, \quad d_{33} = d_{3333}, \quad d_{45} = d_{2313}, \quad d_{66} = d_{1212}, \quad \text{etc.} \tag{3.1}$$

With these notations,

$$\begin{aligned}
Q_{11}(\mathbf{C}) &= d_{11}C_1^2 + d_{66}C_2^2 + d_{55}C_3^2 + 2d_{16}C_1C_2 + 2d_{15}C_1C_3 + 2d_{56}C_2C_3, \\
Q_{22}(\mathbf{C}) &= d_{66}C_1^2 + d_{22}C_2^2 + d_{44}C_3^2 + 2d_{26}C_1C_2 + 2d_{64}C_1C_3 + 2d_{24}C_2C_3, \\
Q_{33}(\mathbf{C}) &= d_{55}C_1^2 + d_{44}C_2^2 + d_{33}C_3^2 + 2d_{45}C_1C_2 + 2d_{35}C_1C_3 + 2d_{34}C_2C_3,
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
Q_{12}(\mathbf{C}) = d_{16}C_1^2 &+ d_{26}C_2^2 + d_{45}C_3^2 \\
&+ (d_{12} + d_{66})C_1C_2 + (d_{14} + d_{56})C_1C_3 + (d_{25} + d_{46})C_2C_3, \\
Q_{23}(\mathbf{C}) = d_{56}C_1^2 &+ d_{24}C_2^2 + d_{34}C_3^2 \\
&+ (d_{25} + d_{46})C_1C_2 + (d_{36} + d_{45})C_1C_3 + (d_{23} + d_{44})C_2C_3, \\
Q_{31}(\mathbf{C}) = d_{15}C_1^2 &+ d_{46}C_2^2 + d_{35}C_3^2 \\
&+ (d_{14} + d_{56})C_1C_2 + (d_{13} + d_{55})C_1C_3 + (d_{36} + d_{45})C_2C_3.
\end{aligned} \tag{3.3}$$

Consider first $\mathbf{C} = (0, 1, i)$. It is isotropic. For this $\mathbf{C}$, conditions (2.13) become $Q_{1k}C_k = 0$ and $iQ_{2k}C_k = Q_{3k}C_k$, which read, explicitly,

$$\begin{aligned}
&(d_{36} + 2d_{45} - d_{26}) - i(d_{25} + 2d_{46} - d_{35}) = 0, \\
&4(d_{34} - d_{24}) + i(d_{22} + d_{33} - 2d_{23} - 4d_{44}) = 0.
\end{aligned} \tag{3.4}$$

Hence,

$$d_{26} = d_{36} + 2d_{45}, \quad d_{35} = d_{25} + 2d_{46}, \quad d_{24} = d_{34}, \quad 4d_{44} = d_{22} + d_{33} - 2d_{23}. \tag{3.5}$$

We then consider in turn $\mathbf{C} = (i, 0, 1)$ and $\mathbf{C} = (1, i, 0)$. These choices yield conditions of the type (3.5), which may be read off from (3.5) on cycling the indices $1 \to 2 \to 3 \to 1$, $4 \to 5 \to 6 \to 4$. The complete set of conditions obtained in this way is

$$\begin{aligned}
&d_{16} = d_{26} = d_{36} + 2d_{45}, \quad d_{35} = d_{15} = d_{25} + 2d_{46}, \quad d_{24} = d_{34} = d_{14} + 2d_{56}, \\
&4d_{44} = d_{22} + d_{33} - 2d_{23}, \; 4d_{55} = d_{33} + d_{11} - 2d_{13}, \; 4d_{66} = d_{11} + d_{22} - 2d_{12}.
\end{aligned} \tag{3.6}$$

We refer to materials whose stiffnesses satisfy these conditions as "special". It follows that "special" materials have at most twelve independent elastic stiffnesses. For instance, they are

$$\begin{array}{cccccc} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ & d_{22} & d_{23} & d_{24} & d_{25} & \bullet \\ & & d_{33} & \bullet & \bullet & d_{36} \\ & & & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet \\ & & & & & \bullet \end{array} \tag{3.7}$$

and the remaining elastic constants (denoted by "$\bullet$" above) are determined from those 12 as

$$\begin{array}{lll} d_{34} = d_{24}, & d_{44} = \frac{1}{4}(d_{22} + d_{33} - 2d_{23}), & d_{45} = \frac{1}{2}(d_{26} - d_{36}), \\ d_{35} = d_{15}, & d_{55} = \frac{1}{4}(d_{11} + d_{33} - 2d_{13}), & d_{56} = \frac{1}{2}(d_{34} - d_{14}) \\ d_{26} = d_{16}, & d_{66} = \frac{1}{4}(d_{11} + d_{22} - 2d_{12}), & d_{46} = \frac{1}{2}(d_{35} - d_{25}). \end{array} \tag{3.8}$$

We note that out of the nine conditions (3.6), six are "structurally invariant" [8] for some rotations of the coordinate system. In particular, if the two conditions

$$d_{16} - d_{26} = d_{11} + d_{22} - 2d_{12} - 4d_{66} = 0, \tag{3.9}$$

are satisfied in the coordinate system linked to the crystallographic axes $(Ox_1x_2x_3)$, then they are also satisfied by the stiffnesses d^*_{ij} obtained from the d_{ij} after any rotation of the coordinate system about the x_3 axis. These invariants are Type 1B in Ting's classification [8]. Similarly, the conditions

$$d_{15} - d_{35} = d_{11} + d_{33} - 2d_{13} - 4d_{55} = 0, \tag{3.10}$$

are invariants under rotation of the coordinate system about the x_2 axis, and

$$d_{24} - d_{34} = d_{22} + d_{33} - 2d_{23} - 4d_{44} = 0, \tag{3.11}$$

are invariants after rotation of the coordinate system about the x_1 axis.

When the relations (3.6) hold, it may be checked by direct calculation, using (3.2), (3.3), and taking into account the relation $C_1^2 + C_2^2 + C_3^2 = 0$, that

$$Q_{ik}(\mathbf{C})C_k = \rho N_L^{-2}(\mathbf{C})C_i, \tag{3.12}$$

for all isotropic bivectors $\mathbf{C}$, with $N_L^{-2}(\mathbf{C})$ given by

$$\rho N_L^{-2}(\mathbf{C}) = \tfrac{1}{2}(d_{11}C_1^2 + d_{22}C_2^2 + d_{33}C_3^2) + 2(d_{16}C_1C_2 + d_{35}C_1C_3 + d_{24}C_2C_3). \tag{3.13}$$

It follows that the relations (3.6) between the elastic stiffnesses are the necessary and sufficient conditions for CPLIPWs to propagate for all choices of isotropic directional bivectors, or, equivalently, for all choices of the polarization plane.

The expression (3.13) may also be written

$$\rho N_L^{-2}(\mathbf{C}) = \mathbf{C} \cdot \mathbf{\Psi}_L \mathbf{C}, \tag{3.14}$$

where $\mathbf{\Psi}_L$ is given by

$$\mathbf{\Psi}_L = \begin{bmatrix} \frac{1}{2}d_{11} & d_{16} & d_{35} \\ d_{16} & \frac{1}{2}d_{22} & d_{24} \\ d_{35} & d_{24} & \frac{1}{2}d_{33} \end{bmatrix}. \tag{3.15}$$

Associated with $\mathbf{\Psi}_L$ is the "$\mathbf{\Psi}_L$-ellipsoid": $\mathbf{x} \cdot (\mathbf{\Psi}_L + p\mathbf{1})\mathbf{x} = 1$, where p is chosen such that $(\mathbf{\Psi}_L + p\mathbf{1})$ is positive definite. If $\mathbf{C}$ is chosen to lie on either plane of central circular section of the $\mathbf{\Psi}_L$-ellipsoid, then $\mathbf{C} \cdot \mathbf{\Psi}_L \mathbf{C} = 0$ and thus $N_L^{-2}(\mathbf{C}) = 0$: the corresponding waves do not propagate.

In the next section, we derive the general structure of the acoustical tensor for "special" materials.

3.2 Acoustical tensor

From the previous section, we know that the acoustical tensor $Q_{ik}(\mathbf{C})$, with $\mathbf{C} \cdot \mathbf{C} = 0$, of "special" materials admits $\rho N_L^{-2}(\mathbf{C})$ given by (3.13) as an eigenvalue. The corresponding eigenvector $\mathbf{C}$ is isotropic, and so [4] this eigenvalue is a double eigenvalue. It follows that the other (simple) eigenvalue $\rho N_*^{-2}(\mathbf{C})$, say, is given by $\rho N_*^{-2}(\mathbf{C}) = \mathrm{tr}\, \mathbf{Q}(\mathbf{C}) - 2\rho N_L^{-2}(\mathbf{C})$, that is,

$$\begin{aligned} \rho N_*^{-2}(\mathbf{C}) = (d_{55} + d_{66})C_1^2 + (d_{44} + d_{66})C_2^2 + (d_{44} + d_{55})C_3^2 \\ + 2d_{45}C_1C_2 + 2d_{46}C_1C_3 + 2d_{56}C_2C_3. \end{aligned} \tag{3.16}$$

The expression (3.16) may also be written

$$\rho N_*^{-2}(\mathbf{C}) = \mathbf{C} \cdot \mathbf{\Psi}_T \mathbf{C}, \tag{3.17}$$

where $\mathbf{\Psi}_T$ is given by

$$\mathbf{\Psi}_T = \begin{bmatrix} d_{55}+d_{66} & d_{45} & d_{46} \\ d_{45} & d_{44}+d_{66} & d_{56} \\ d_{46} & d_{56} & d_{44}+d_{55} \end{bmatrix}. \tag{3.18}$$

Associated with $\mathbf{\Psi}_T$ is the "$\mathbf{\Psi}_T$-ellipsoid": $\mathbf{x}\cdot(\mathbf{\Psi}_T + p\mathbf{1})\mathbf{x} = 1$, where p is chosen such that $(\mathbf{\Psi}_T + p\mathbf{1})$ is positive definite. If $\mathbf{C}$ is chosen to lie on either plane of central circular section of the $\mathbf{\Psi}_T$-ellipsoid, then $\mathbf{C}\cdot\mathbf{\Psi}_T\mathbf{C} = 0$ and so $N_*^{-2}(\mathbf{C}) = 0$: there is no corresponding transverse circularly polarized propagating wave.

Now we compute the components of the matrix $\mathbf{\Gamma}(\mathbf{C}) := \mathbf{Q}(\mathbf{C}) - \rho N_*^{-2}(\mathbf{C})\mathbf{1}$. We find, using the conditions (3.6) and $C_1^2 + C_2^2 + C_3^2 = 0$, that

$$\begin{aligned} \Gamma_{11}(\mathbf{C}) &= (\mu+\nu-\lambda)C_1^2 + 2\gamma C_1C_2 + 2\beta C_1C_3, \\ \Gamma_{22}(\mathbf{C}) &= (\nu+\lambda-\mu)C_2^2 + 2\alpha C_2C_3 + 2\gamma C_1C_2, \\ \Gamma_{33}(\mathbf{C}) &= (\lambda+\mu-\nu)C_3^2 + 2\beta C_1C_3 + 2\alpha C_2C_3, \end{aligned} \tag{3.19}$$

and that

$$\begin{aligned} \Gamma_{12}(\mathbf{C}) &= -\gamma C_3^2 + \nu C_1C_2 + \alpha C_1C_3 + \beta C_2C_3, \\ \Gamma_{23}(\mathbf{C}) &= -\alpha C_1^2 + \lambda C_2C_3 + \beta C_1C_2 + \gamma C_1C_3, \\ \Gamma_{31}(\mathbf{C}) &= -\beta C_2^2 + \mu C_1C_3 + \gamma C_2C_3 + \alpha C_1C_2, \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} &\lambda := \tfrac{1}{2}(d_{22}+d_{33}-2d_{44}), \quad &&\mu := \tfrac{1}{2}(d_{11}+d_{33}-2d_{55}), \quad &&\nu := \tfrac{1}{2}(d_{11}+d_{22}-2d_{66}), \\ &\alpha := d_{14}+d_{56}, &&\beta := d_{25}+d_{46}, &&\gamma := d_{36}+d_{45}. \end{aligned} \tag{3.21}$$

Let $\mathbf{M}$ be the real symmetric matrix defined as

$$\mathbf{M} := \begin{bmatrix} \lambda & -\gamma & -\beta \\ -\gamma & \mu & -\alpha \\ -\beta & -\alpha & \nu \end{bmatrix}. \tag{3.22}$$

It can be checked that

$$\mathbf{\Gamma}(\mathbf{C}) = (\lambda+\mu+\nu)\mathbf{C}\otimes\mathbf{C} - \mathbf{MC}\otimes\mathbf{C} - \mathbf{C}\otimes\mathbf{MC}. \tag{3.23}$$

Alternatively, introducing $\hat{\mathbf{M}} := \frac{1}{2}(\lambda + \mu + \nu)\mathbf{1} - \mathbf{M}$, the matrix $\mathbf{\Gamma}(\mathbf{C})$ may be written as

$$\mathbf{\Gamma}(\mathbf{C}) = \hat{\mathbf{M}}\mathbf{C} \otimes \mathbf{C} + \mathbf{C} \otimes \hat{\mathbf{M}}\mathbf{C}, \tag{3.24}$$

where, explicitly,

$$\hat{\mathbf{M}} = \begin{bmatrix} \frac{1}{2}(d_{11} + d_{44} - d_{55} - d_{66}) & d_{36} + d_{45} & d_{25} + d_{46} \\ d_{36} + d_{45} & \frac{1}{2}(d_{22} + d_{55} - d_{44} - d_{66}) & d_{14} + d_{56} \\ d_{25} + d_{46} & d_{14} + d_{56} & \frac{1}{2}(d_{33} + d_{66} - d_{44} - d_{55}) \end{bmatrix}. \tag{3.25}$$

Hence, noting that $\mathbf{C} \cdot \mathbf{C} = 0$ and recalling the definition of $\mathbf{\Gamma}(\mathbf{C})$, we find that the acoustical tensor may be put in the form,

$$\mathbf{Q}(\mathbf{C}) = \rho N_*^{-2}(\mathbf{C})\mathbf{1} + \hat{\mathbf{M}}\mathbf{C} \otimes \mathbf{C} + \mathbf{C} \otimes \hat{\mathbf{M}}\mathbf{C}. \tag{3.26}$$

This decomposition of the acoustical tensor shows directly that the isotropic bivector $\mathbf{C}$ is an eigenvector of $\mathbf{Q}(\mathbf{C})$, whose eigenvalue $\rho N_L^{-2}(\mathbf{C})$ given by (3.13), can equivalently be written

$$\rho N_L^{-2}(\mathbf{C}) = \rho N_*^{-2}(\mathbf{C}) + \mathbf{C} \cdot \hat{\mathbf{M}}\mathbf{C}. \tag{3.27}$$

Also, the eigenvector corresponding to the eigenvalue $\rho N_*^{-2}(\mathbf{C})$ is $\mathbf{C} \times \hat{\mathbf{M}}\mathbf{C}$. We call the corresponding wave the "transverse" inhomogeneous plane wave, because its amplitude bivector $\mathbf{A}$ is orthogonal to $\mathbf{C}$: $\mathbf{C} \cdot \mathbf{A} = 0$. In general it is elliptically polarized because $(\mathbf{C} \times \hat{\mathbf{M}}\mathbf{C}) \cdot (\mathbf{C} \times \hat{\mathbf{M}}\mathbf{C}) = -(\mathbf{C} \cdot \hat{\mathbf{M}}\mathbf{C})^2 \neq 0$. Of course $\mathbf{C} \cdot [\mathbf{C} \times \hat{\mathbf{M}}\mathbf{C}] = 0$, which means [4] that the orthogonal projection, upon the plane of $\mathbf{C}$, of the ellipse associated with the amplitude bivector $\mathbf{C} \times \hat{\mathbf{M}}\mathbf{C}$, is a circle.

Let $\tilde{\mathbf{C}}$ be a choice of $\mathbf{C}$ for which $\tilde{\mathbf{C}} \cdot \hat{\mathbf{M}}\tilde{\mathbf{C}} = \tilde{\mathbf{C}} \cdot \tilde{\mathbf{C}} = 0$, so that $\tilde{\mathbf{C}}$ is parallel to $\tilde{\mathbf{C}} \times \hat{\mathbf{M}}\tilde{\mathbf{C}}$ and there is a triple eigenvalue $\rho N_L^{-2}(\tilde{\mathbf{C}}) = \rho N_*^{-2}(\tilde{\mathbf{C}})$. This special case occurs when the plane of $\tilde{\mathbf{C}}$ is one of the two planes of central circular section of the the "$\hat{\mathbf{M}}$-ellipsoid": $\mathbf{x} \cdot (\hat{\mathbf{M}} + p\mathbf{1})\mathbf{x} = 1$, where p is chosen such that $(\hat{\mathbf{M}} + p\mathbf{1})$ is positive definite.

Remark: A relationship between the scalar slownesses of certain waves

The form (3.27) of the relation triggers the following remark. Let the matrix $\hat{\mathbf{M}}$ have eigenvalues p_1, p_2, p_3 and corresponding unit orthogonal eigenvectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ so that

$$\hat{\mathbf{M}} = p_1\mathbf{a}_1 \otimes \mathbf{a}_1 + p_2\mathbf{a}_2 \otimes \mathbf{a}_2 + p_3\mathbf{a}_3 \otimes \mathbf{a}_3, \quad \mathbf{a}_i \cdot \mathbf{a}_j = \delta_{ij}. \tag{3.28}$$

Then consider the following isotropic bivectors $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$,

$$\mathbf{C}_1 := \mathbf{a}_2 + i\mathbf{a}_3, \quad \mathbf{C}_2 := \mathbf{a}_3 + i\mathbf{a}_1, \quad \mathbf{C}_3 := \mathbf{a}_1 + i\mathbf{a}_2. \tag{3.29}$$

Clearly,

$$\mathbf{C}_1 \cdot \hat{\mathbf{M}}\mathbf{C}_1 = p_2 - p_3, \quad \mathbf{C}_2 \cdot \hat{\mathbf{M}}\mathbf{C}_2 = p_3 - p_1, \quad \mathbf{C}_3 \cdot \hat{\mathbf{M}}\mathbf{C}_3 = p_1 - p_2, \tag{3.30}$$

so that

$$\sum_{i=1}^{3} \mathbf{C}_i \cdot \hat{\mathbf{M}}\mathbf{C}_i = 0. \tag{3.31}$$

Then, from (3.27) and (3.31), we have the relation

$$\sum_{i=1}^{3} N_L^{-2}(\mathbf{C}_i) = \sum_{i=1}^{3} N_*^{-2}(\mathbf{C}_i), \tag{3.32}$$

for the $\mathbf{C}_i$ given by (3.29).

Example

To illustrate the results of this section, we work out a simple example. Let $\mathbf{C} = \mathbf{i} + i\mathbf{j}$. Then the corresponding acoustical tensor $\mathbf{Q}(\mathbf{i} + i\mathbf{j})$ is

$$\begin{bmatrix} d_{11} - d_{66} + 2id_{16} & i(d_{66} + d_{12}) & d_{15} - d_{46} + i(d_{56} + d_{14}) \\ i(d_{66} + d_{12}) & d_{66} - d_{22} + 2id_{26} & -(d_{56} + d_{14}) + i(d_{15} - d_{46}) \\ d_{15} - d_{46} + i(d_{56} + d_{14}) & -(d_{56} + d_{14}) + i(d_{15} - d_{46}) & d_{55} - d_{44} + 2id_{45} \end{bmatrix}. \tag{3.33}$$

Computing $\mathbf{Q}(\mathbf{C})\mathbf{C}$ and using the conditions (3.6), we find that $\mathbf{C}$ is indeed an eigenvector of the acoustical tensor, with eigenvalue given by (3.13),

$$\mathbf{Q}(\mathbf{C})\mathbf{C} = \rho N_L^{-2}\mathbf{C}, \quad \rho N_L^{-2} = \tfrac{1}{2}(d_{11} - d_{22}) + 2id_{16}. \tag{3.34}$$

Further, computing $\mathbf{C} \times \hat{\mathbf{M}}\mathbf{C}$ and then $\mathbf{Q}(\mathbf{C})(\mathbf{C} \times \hat{\mathbf{M}}\mathbf{C})$ and using the conditions (3.6), we find that $\mathbf{C} \times \hat{\mathbf{M}}\mathbf{C}$ is also an eigenvector of the acoustical tensor, with eigenvalue now given by (3.16),

$$\mathbf{Q}(\mathbf{C})(\mathbf{C} \times \hat{\mathbf{M}}\mathbf{C}) = \rho N_*^{-2}(\mathbf{C} \times \hat{\mathbf{M}}\mathbf{C}), \quad \rho N_*^{-2} = d_{55} - d_{44} + 2id_{45}. \tag{3.35}$$

In this simple example, we prescribed the normal to the plane of the isotropic slowness bivector (or equivalently, to the plane of the amplitude bivector) to be $\mathbf{k}$. Then, we chose $\mathbf{C}$ to be $\mathbf{i} + i\mathbf{j}$, leading to a complex eigenvalue. In the next section, we show that it is always possible to choose $\mathbf{C}$ such that the corresponding eigenvalue is a real positive number.

Remark: General form of the acoustical tensor for the "special" materials

Using the relations (3.6) in the expressions $Q_{ij}(\mathbf{C})$ given by (3.2) and (3.3), it may be seen that without any restrictions on $\mathbf{C}$, the acoustical tensor $\mathbf{Q}(\mathbf{C})$ for the "special" materials may be written as

$$\mathbf{Q}(\mathbf{C}) = \rho N_*^{-2}(\mathbf{C})\mathbf{1} + \hat{\mathbf{M}}\mathbf{C} \otimes \mathbf{C} + \mathbf{C} \otimes \hat{\mathbf{M}}\mathbf{C} + (\mathbf{C} \cdot \mathbf{C})\boldsymbol{\Delta}. \tag{3.36}$$

Here the expression for $\rho N_*^{-2}(\mathbf{C})$ is given by (3.16) and $\boldsymbol{\Delta}$ is defined by

$$\boldsymbol{\Delta} = \begin{bmatrix} -d_{44} & d_{45} & d_{46} \\ d_{45} & -d_{55} & d_{56} \\ d_{46} & d_{56} & -d_{66} \end{bmatrix}. \tag{3.37}$$

Comparing (3.18) and (3.37), we note that $\boldsymbol{\Psi}_T = \boldsymbol{\Delta} - (\text{tr }\boldsymbol{\Delta})\mathbf{1}$, so that by (3.17),

$$\rho N_*^{-2}(\mathbf{C}) = \mathbf{C} \cdot \boldsymbol{\Delta}\mathbf{C} - (\text{tr }\boldsymbol{\Delta})\mathbf{C} \cdot \mathbf{C}. \tag{3.38}$$

It follows that

$$\mathbf{Q}(\mathbf{C}) = [\mathbf{C} \cdot \boldsymbol{\Delta}\mathbf{C} - (\text{tr }\boldsymbol{\Delta})\mathbf{C} \cdot \mathbf{C}]\mathbf{1} + \hat{\mathbf{M}}\mathbf{C} \otimes \mathbf{C} + \mathbf{C} \otimes \hat{\mathbf{M}}\mathbf{C} + (\mathbf{C} \cdot \mathbf{C})\boldsymbol{\Delta}, \tag{3.39}$$

which is an expression for the acoustical tensor $\mathbf{Q}(\mathbf{C})$ for "special" materials, given in terms of only two matrices, $\boldsymbol{\Delta}$ and $\hat{\mathbf{M}}$.

We note for "special" materials that d_{ijkl} may be written

$$\begin{aligned} d_{ijkl} = {} & \Delta_{jl}\delta_{ik} + \Delta_{ik}\delta_{jl} + \Delta_{il}\delta_{jk} + \Delta_{jk}\delta_{il} - \Delta_{ij}\delta_{kl} - \Delta_{kl}\delta_{ij} \\ & - (\text{tr }\boldsymbol{\Delta})(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il} - \delta_{ij}\delta_{kl}) + \hat{M}_{ij}\delta_{kl} + \hat{M}_{kl}\delta_{ij}. \end{aligned} \tag{3.40}$$

Remark: Homogeneous plane waves in "special" materials

For *homogeneous* plane waves propagating in the direction $\mathbf{n}$ in the "special" materials, let $\mathbf{u} = \mathbf{A} \exp i\omega(S\mathbf{n} \cdot \mathbf{x} - t)$. The corresponding propagation condition is then

$$\mathbf{Q}(\mathbf{n})\mathbf{A} = \rho S^2 \mathbf{A}, \tag{3.41}$$

where $\mathbf{Q}(\mathbf{n})$ is given by (3.39) with n_1, n_2, n_3 replacing C_1, C_2, C_3, respectively. Thus

$$\mathbf{Q}(\mathbf{n}) = [\mathbf{n} \cdot \boldsymbol{\Delta}\mathbf{n} - (\text{tr }\boldsymbol{\Delta})]\mathbf{1} + \hat{\mathbf{M}}\mathbf{n} \otimes \mathbf{n} + \mathbf{n} \otimes \hat{\mathbf{M}}\mathbf{n} + \boldsymbol{\Delta}. \tag{3.42}$$

We note a special solution.

Let the Hamiltonian decomposition of $\mathbf{\Delta}$ be given by [3]

$$\mathbf{\Delta} = \mu \mathbf{1} + \kappa(\mathbf{h}^+ \otimes \mathbf{h}^- + \mathbf{h}^- \otimes \mathbf{h}^+), \tag{3.43}$$

where μ, κ are constants and $\mathbf{h}^\pm$ are constant unit vectors. Choose $\mathbf{n} = \mathbf{n}^*$ such that

$$(\mathbf{n}^* \times \hat{\mathbf{M}}\mathbf{n}^*) \times (\mathbf{h}^+ \times \mathbf{h}^-) = \mathbf{0}. \tag{3.44}$$

(It is always possible to do this.) A solution for a homogeneous plane wave propagating along $\mathbf{n}^*$ is

$$\mathbf{u} = (\mathbf{h}^+ \times \mathbf{h}^-) \exp i\omega(N^{-1}\mathbf{n}^* \cdot \mathbf{x} - t), \quad \text{where} \quad \rho N^{-2} = \mu + \mathbf{n}^* \cdot \mathbf{\Delta}\mathbf{n}^* - (\text{tr } \mathbf{\Delta}). \tag{3.45}$$

4 Description of the waves

Before proceeding, we note that the two equations (3.13) and (3.16), giving the complex scalar slowness of the longitudinal circularly polarized wave and of the transverse elliptically polarized wave, respectively, have the same form,

$$\mathbf{C} \cdot \mathbf{\Psi}\mathbf{C} = \rho N^{-2}, \quad \mathbf{C} \cdot \mathbf{C} = 0, \tag{4.1}$$

where $\mathbf{\Psi}$ is a real symmetric tensor given by $\mathbf{\Psi} = \mathbf{\Psi}_L$ (see (3.15)) for the longitudinal wave and by $\mathbf{\Psi} = \mathbf{\Psi}_T$ (see (3.18)) for the transverse wave. Accordingly, we determine the details of the slowness of the corresponding wave solutions. The results may be adapted either to the longitudinal wave or to the transverse wave by replacing $\mathbf{\Psi}$ with either $\mathbf{\Psi}_L$ or $\mathbf{\Psi}_T$.

4.1 Construction of the slowness bivector

Let Ψ_1, Ψ_2, Ψ_3 be the eigenvalues of $\mathbf{\Psi}$ and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ the corresponding orthogonal unit eigenvectors. We assume that the eigenvalues are ordered as $\Psi_1 > \Psi_2 > \Psi_3$. We note that the pair (4.1) is equivalent to the pair

$$\mathbf{C} \cdot (\mathbf{\Psi} + p\mathbf{1})\mathbf{C} = \rho N^{-2}, \quad \mathbf{C} \cdot \mathbf{C} = 0, \tag{4.2}$$

where p is an arbitrary constant. By choosing p suitably large and positive, so that $\Psi_3 + p > 0$, we may define the positive definite matrix $(\mathbf{\Psi} + p\mathbf{1})$ and associate the ellipsoid

$$\mathbf{x} \cdot (\mathbf{\Psi} + p\mathbf{1})\mathbf{x} = 1, \tag{4.3}$$

with $\mathbf{\Psi}$. We call this the "$\mathbf{\Psi}$-ellipsoid". The planes of central circular section of the $\mathbf{\Psi}$-ellipsoid have unit normals $\mathbf{h}^{\pm}$ given by [3]

$$(\Psi_1 - \Psi_3)^{\frac{1}{2}}\mathbf{h}^{\pm} = (\Psi_1 - \Psi_2)^{\frac{1}{2}}\mathbf{e}_1 \pm (\Psi_2 - \Psi_3)^{\frac{1}{2}}\mathbf{e}_3. \tag{4.4}$$

We call these normals $\mathbf{h}^{\pm}$, the "optic axes" of the $\mathbf{\Psi}$-ellipsoid and note that they are independent of p.

To describe the slownesses of the waves corresponding to (4.1), we recall that

$$\mathbf{S} = N\mathbf{C}, \quad \mathbf{C} = \hat{\mathbf{m}} + i\hat{\mathbf{n}}, \quad \hat{\mathbf{m}}\cdot\hat{\mathbf{m}} = \hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = 1, \quad \hat{\mathbf{m}}\cdot\hat{\mathbf{n}} = 0. \tag{4.5}$$

Here $\mathbf{C}$ is such that $\mathbf{C}\cdot\mathbf{C} = 0$ and thus $\mathbf{S}\cdot\mathbf{S} = 0$. Also $(\hat{\mathbf{m}}, \hat{\mathbf{n}})$ is *any* pair of orthogonal unit vectors in the plane of $\mathbf{S}$, or equivalently in the plane with unit normal $\mathbf{a} := \hat{\mathbf{m}} \times \hat{\mathbf{n}}$. In (4.5), N is a scalar to be determined from the equations (4.1).

Because we are at liberty to choose for $(\hat{\mathbf{m}}, \hat{\mathbf{n}})$ any orthogonal unit pair in the plane with normal $\mathbf{a}$, we choose $(\hat{\mathbf{m}}, \hat{\mathbf{n}})$ along the principal axes of the elliptical section of the $\mathbf{\Psi}$-ellipsoid by the central plane $\mathbf{a}\cdot\mathbf{x} = 0$. Specifically, we take $\hat{\mathbf{m}}$ as the unit vector along the minor axis of this ellipse and $\hat{\mathbf{n}}$ along the major axis. In that event,

$$\hat{\mathbf{m}}\cdot\mathbf{\Psi}\hat{\mathbf{n}} = 0, \tag{4.6}$$

and using (4.5) and (4.6), the pair (4.1) give

$$\rho N^{-2} = \hat{\mathbf{m}}\cdot\mathbf{\Psi}\hat{\mathbf{m}} - \hat{\mathbf{n}}\cdot\mathbf{\Psi}\hat{\mathbf{n}} > 0, \tag{4.7}$$

in general, so that N^{-1} is purely real: $N^{-1} = v$, say. Also,

$$\mathbf{S}^{+} = v^{-1}\hat{\mathbf{m}}, \quad \mathbf{S}^{-} = v^{-1}\hat{\mathbf{n}}, \tag{4.8}$$

so that the planes of constant phase (amplitude) are: $\hat{\mathbf{m}}\cdot\mathbf{x}$ = constant ($\hat{\mathbf{n}}\cdot\mathbf{x}$ = constant).

Of course, if $\hat{\mathbf{m}}\cdot\mathbf{\Psi}\hat{\mathbf{m}} = \hat{\mathbf{n}}\cdot\mathbf{\Psi}\hat{\mathbf{n}}$, so that the radii to the $\mathbf{\Psi}$-ellipsoid along the orthogonal unit vectors are equal, then the plane $\mathbf{a}\cdot\mathbf{x} = 0$ is a plane of central circular section of the $\mathbf{\Psi}$-ellipsoid, and from (4.7) there is no propagating solution: $\rho N^{-2} = 0$.

Thus, in general, the slowness bivectors corresponding to (4.1) are obtained by first determining the central elliptical section of the $\mathbf{\Psi}$-ellipsoid by

the plane $\mathbf{a} \cdot \mathbf{x} = 0$. Then $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ are chosen along the principal axes of the ellipse, and $\mathbf{S}^+$ and $\mathbf{S}^-$ are given by (4.7) (4.8).

To complete the picture we recall the results [3] for the determination of the principal axes of the central elliptical section of the $\mathbf{\Psi}$-ellipsoid by the plane $\mathbf{a} \cdot \mathbf{x} = 0$.

In the determination there are three cases to be considered:

Case(i) Normal $\mathbf{a}$ not coplanar with the optic axes.

Case(ii) Normal $\mathbf{a}$ coplanar with the optic axes but not parallel to either optic axis.

Case(iii) Normal $\mathbf{a}$ parallel to an optic axis.

We can dispose of Case (iii) immediately because we have just seen that there is no propagating circularly polarized solution when $\mathbf{a} \cdot \mathbf{x} = 0$ is a plane of central circular section of the $\mathbf{\Psi}$-ellipsoid.

In Case (i), $\mathbf{a} \cdot \mathbf{h}^+ \times \mathbf{h}^- \neq 0$, it has been shown [3] that $\hat{\mathbf{m}}$, $\hat{\mathbf{n}}$ are in the direction of $\mathbf{r}^\pm$ given by

$$\mathbf{r}^\pm = [\mathbf{h}^+ - (\cos\phi^+)\mathbf{a}]/(\sin\phi^+) \pm [\mathbf{h}^- - (\cos\phi^-)\mathbf{a}]/(\sin\phi^-), \tag{4.9}$$

where $\phi^\pm$ is the angle between $\mathbf{a}$ and the optic axis $\mathbf{h}^\pm$. Essentially, $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ are along the internal and external bisectors of the angle between the orthogonal projections of $\mathbf{h}^+$ and $\mathbf{h}^-$ onto the plane $\mathbf{a} \cdot \mathbf{x} = 0$ (this is the Fresnel construction of Optics.)

In Case (ii), $\mathbf{a} \cdot \mathbf{h}^+ \times \mathbf{h}^- = 0$, $\mathbf{a} \times \mathbf{h}^\pm \neq \mathbf{0}$, it has been shown [3] that

$$\hat{\mathbf{m}} = [\mathbf{h}^+ - (\cos\phi^+)\mathbf{a}]/(\sin\phi^+), \quad \hat{\mathbf{n}} = (\mathbf{h}^+ \times \mathbf{a})/(\sin\phi^+). \tag{4.10}$$

Essentially, $\hat{\mathbf{m}}$ is along the orthogonal projection of $\mathbf{h}^+$ (or $\mathbf{h}^-$) onto the plane $\mathbf{a} \cdot \mathbf{x} = 0$, and $\hat{\mathbf{n}}$ is orthogonal to that plane.

This means that as the direction of the unit normal $\mathbf{a}$ is varied, the corresponding $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ are given by (4.9) and (4.10). Also, as shown by Boulanger and Hayes [3], in Cases (i) and (ii), $\hat{\mathbf{m}} \cdot \mathbf{\Psi}\hat{\mathbf{m}} - \hat{\mathbf{n}} \cdot \mathbf{\Psi}\hat{\mathbf{n}} = (\Psi_1 - \Psi_3)\sin\phi^+ \sin\phi^-$, so that from (4.7),

$$\rho N^{-2} = \rho v^2 = (\Psi_1 - \Psi_3)\sin\phi^+ \sin\phi^-, \tag{4.11}$$

where $\phi^\pm$ are the angles that the normal $\mathbf{a}$ to the plane of $\mathbf{S}$ makes with the optic axes.

We summarize the situation.

To determine all $\mathbf{S}^+$ and $\mathbf{S}^-$ corresponding to (4.1), the "optics axes" $\mathbf{h}^\pm$ are first determined. Then the plane of $\mathbf{C} = \hat{\mathbf{m}} + i\hat{\mathbf{n}}$ is chosen; it has unit

normal $\mathbf{a}$. If $\mathbf{a}$ is not coplanar with the two optic axes, then the corresponding $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ are in the directions of $\mathbf{r}^{\pm}$ given by (4.9) and N is given by (4.7). If $\mathbf{a}$ is coplanar with the optic axes $\mathbf{h}^{\pm}$ but not along either of them, then the corresponding $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ are given by (4.10) and N is given by (4.11). If $\mathbf{a}$ is along $\mathbf{h}^{+}$ or $\mathbf{h}^{-}$, there is no propagating wave.

4.2 Example: Crystal with a plane of symmetry

To illustrate the method described above, we take a "special" material with a symmetry plane at $x_3 = 0$, say. Then the tensor $\mathbf{\Psi}_L$ defined in (3.15) for the longitudinal wave is given by

$$\mathbf{\Psi}_L = \begin{bmatrix} \frac{1}{2}d_{11} & d_{16} & 0 \\ d_{16} & \frac{1}{2}d_{22} & 0 \\ 0 & 0 & \frac{1}{2}d_{33} \end{bmatrix}. \tag{4.12}$$

Its eigenvalues are

$$\Psi_{1,3} = \tfrac{1}{4}\left[d_{11} + d_{22} \pm \sqrt{(d_{11} - d_{22})^2 + 16d_{16}^2}\right], \quad \Psi_2 = \tfrac{1}{2}d_{33}. \tag{4.13}$$

Here we assume that the stiffnesses of the material are such that these eigenvalues are ordered $\Psi_1 > \Psi_2 > \Psi_3$. The corresponding unit eigenvectors are

$$\mathbf{e}_1 = \frac{1}{\delta}\begin{bmatrix} \frac{1}{2}d_{22} - \Psi_1 \\ -d_{16} \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{e}_3 = \frac{1}{\delta}\begin{bmatrix} -d_{16} \\ \frac{1}{2}d_{11} - \Psi_3 \\ 0 \end{bmatrix}, \tag{4.14}$$

where δ is the positive quantity given by

$$\delta^2 = \tfrac{1}{8}(d_{11} - d_{22})^2 + 2d_{16}^2 + \tfrac{1}{4}(d_{11} - d_{22})\sqrt{(d_{11} - d_{22})^2 + 16d_{16}^2}. \tag{4.15}$$

With this choice, the optic axes defined by (4.4) lie in the symmetry plane $x_3 = 0$.

For simplicity, we now focus on waves polarized in the symmetry plane, that is, we choose the normal $\mathbf{a}$ to the plane of $\mathbf{S}$ to be along $\mathbf{e}_2$. Then $\mathbf{a}$ is obviously not coplanar with the optic axes (Case (i) of the previous subsection). Specifically, the angles between $\mathbf{a}$ and the optic axes are $\phi^{+} =$

$\phi^- = \pi/2$. It follows from (4.9) that $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ are in the directions of $\mathbf{h}^+ \pm \mathbf{h}^-$, i.e.

$$\hat{\mathbf{m}} = \mathbf{e}_1, \quad \hat{\mathbf{n}} = \mathbf{e}_3. \tag{4.16}$$

We conclude that the following CPLIPW may propagate in a monoclinic crystal with symmetry plane at $x_3 = 0$ and with stiffnesses satisfying (3.6),

$$\mathbf{u} = \mathrm{e}^{-k\mathbf{e}_3 \cdot \mathbf{x}} \{ \mathbf{e}_1 \cos k(\mathbf{e}_1 \cdot \mathbf{x} - vt) - \mathbf{e}_3 \sin k(\mathbf{e}_1 \cdot \mathbf{x} - vt) \}, \tag{4.17}$$

where the orthogonal unit vectors $\mathbf{e}_1$, $\mathbf{e_3}$ are defined in (4.14), k is an arbitrary real wave number, and the real speed v is given by

$$\rho v^2 = \Psi_1 - \Psi_3 = \tfrac{1}{2}\sqrt{(d_{11} - d_{22})^2 + 16 d_{16}^2}. \tag{4.18}$$

5 Crystals with symmetries

In this section we investigate how the conditions (3.6) for an anisotropic crystal to admit a CPLIPW for all choices of the polarization plane are affected when the crystal presents certain symmetries.

5.1 Monoclinic crystals

Here we consider crystals with a plane of symmetry, at $x_3 = 0$ say. For that class of materials,

$$d_{14} = d_{15} = d_{24} = d_{25} = d_{34} = d_{35} = d_{46} = d_{56} = 0. \tag{5.1}$$

It follows that four out of the nine equations (3.6) reduce to trivial identities, automatically satisfied. The nine conditions (3.6) reduce to a set of *five* equations,

$$\begin{aligned} &d_{16} = d_{26} = d_{36} + 2d_{45}, \\ &4d_{44} = d_{22} + d_{33} - 2d_{23}, \; 4d_{55} = d_{33} + d_{11} - 2d_{13}, \; 4d_{66} = d_{11} + d_{22} - 2d_{12}. \end{aligned} \tag{5.2}$$

5.2 Orthorhombic crystals

Orthorhombic crystals possess three symmetry planes, at $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. In addition to (5.1), the relations

$$d_{16} = d_{26} = d_{36} = d_{45} = 0, \tag{5.3}$$

also hold. The set of nine conditions (3.6) now reduces to a set of *three* equations,

$$4d_{44} = d_{22}+d_{33}-2d_{23}, \; 4d_{55} = d_{33}+d_{11}-2d_{13}, \; 4d_{66} = d_{11}+d_{22}-2d_{12}. \quad (5.4)$$

5.3 Trigonal, tetragonal, and cubic crystals

For *trigonal crystals*, $d_{24} = -d_{14} = -d_{56} \neq 0$, and one of the conditions (3.6), namely: $d_{24} = d_{14} + 2d_{56}$, cannot be satisfied.

For *tetragonal crystals*, $d_{11} = d_{22}$, $d_{13} = d_{23}$, $d_{44} = d_{55}$, and the condition: $4d_{66} = d_{11} + d_{22} - 2d_{12}$ reduces to: $d_{66} = (d_{11} - d_{12})/2$, which would mean that the crystal is in fact hexagonal (transversally isotropic).

For *cubic crystals*, $d_{11} = d_{22} = d_{33}$, $d_{12} = d_{23} = d_{13}$, $d_{44} = d_{55} = d_{66}$, and the condition: $4d_{66} = d_{11} + d_{22} - 2d_{12}$ reduces to: $d_{66} = (d_{11} - d_{12})/2$, which would mean that the material is in fact isotropic.

We conclude that there are no trigonal, no tetragonal, and no cubic crystals in which CPLIPWs may propagate for all orientations of the slowness plane.

5.4 Hexagonal crystals

For hexagonal crystals, the following relations hold for the stiffnesses,

$$d_{11} = d_{22}, \quad d_{13} = d_{23}, \quad d_{44} = d_{55}, \quad d_{66} = (d_{11} - d_{12})/2, \qquad (5.5)$$

in addition to (5.1) and (5.3). The nine equations (3.6) reduce to a *single* equation,

$$d_{44} = \tfrac{1}{4}(d_{11} + d_{33} - 2d_{13}). \qquad (5.6)$$

5.5 Isotropic materials

For isotropic materials, the following relations hold for the stiffnesses,

$$d_{11} = d_{22} = d_{33}, \quad d_{12} = d_{23} = d_{13}, \quad d_{44} = d_{55} = d_{66} = (d_{11} - d_{12})/2, \quad (5.7)$$

in addition to (5.1) and (5.3). Then, the nine equations (3.6) are all identically satisfied. However, the propagation condition (3.13), giving the complex scalar slowness N_L now simplifies to

$$\rho N_L^{-2}(\mathbf{C}) = \tfrac{1}{2}d_{11}(C_1^2 + C_2^2 + C_3^2) = 0, \qquad (5.8)$$

for isotropic slownesses. It follows that CPLIPWs may not propagate in an isotropic material, for any choice of isotropic slowness.

On the other hand, this analysis shows that "longitudinal" *static* exponential solutions with an isotropic slowness bivector always exist for linear isotropic elastic materials. This result may be checked directly in the following manner. Recall the classical equations of equilibrium of an isotropic material,

$$(\lambda+\mu)u_{j,ij}+\mu u_{i,jj}=0, \tag{5.9}$$

where λ and μ are the Lamé constants. Then it is easy to check that the field

$$u_i=S_i e^{i\omega S_j x_j}, \quad S_k S_k=0, \tag{5.10}$$

is indeed an exact solution.

We sum up the situation. There are no isotropic, cubic, trigonal, or tetragonal elastic crystals such that CPLIPWs may propagate in every plane. However, the propagation of CPLIPWs is theoretically possible for all planes of some triclinic, monoclinic, or orthorhombic elastic crystals provided some relations among the elastic stiffnesses are satisfied. We note that there is unfortunately insufficient data for such crystals available at present to enable us present an explicit example, but we recall that the values of the elastic stiffnesses change with pressure, temperature, prestress, etc. and that they may consequently be adjusted to produce an adequate crystal.

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